ISS-Lyapunov Functions for Output Feedback Sliding Modes

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***Abstract*— In this paper we revisit the problem of stabilizing a triple integrator using a control that depends on the signs of the state variables. For a more general class of linear systems it is shown that the stabilization by sign feedback is possible, depending on some properties of the system's matrix. The conditions for the stability are established by means of linear matrix inequalities. For the triple integrator, the domain of stability is evaluated. Also, the control law is augmented by a linear feedback and the stability properties for this case, checked. The results are illustrated by numerical experiments for a chain of integrators of third order.**

I. Introduction

In the last three decades the sliding mode controllers have gained a lot of attention from the control community due to their robustness and finite-time convergence properties [2], [6], [8]. The benchmark problem consists in the stabilization of a chain of integrators. In the well-known case of a single integrator, $\dot{x} = u + w(t)$, it can be finite-time stabilized by a sign controller of the type $u = -k\,\mathrm{sign}(x)$, where the gain k is chosen adequately [9]. For the second order system, $\dot{x}_1 = x_2, \dot{x}_2 = u + w(t)$, the Twisting controller $\left(u = -k_1\mathrm{sign}(x_1) - k_2\mathrm{sign}(x_2)\right)$ was introduced in [5] with the same properties. The only knowledge of the system necessary to implement any of these controllers is the sign of their state variables. This makes them a suitable alternative in systems where a state feedback is not feasible due to uncertainties in the sensors and where the only truthful information available is precisely the sign of the state variables. In the design of practical applications this is beneficial because it reduces the sensor requirements and therefore diminishes the costs and complexity of the implementation. A natural step to develop the mentioned results would be the establishment of conditions for the stability of a third order chain of integrators using only the signs of its state variables and, further on, a generalization to a system of an arbitrary order. Therefore, some works can be found in the control literature, which investigate this matter for the third order, with different approaches. For example, in [1] it is proposed to switch between two different controllers that use only the signs of the state variables, following a specific logic. This scheme was later generalized for any order in [4]. In [7] it was proven that, for certain initial conditions, the trajectories of the triple integrator, in closed loop with the triple sign controller, and an appropriate choice of gains, will converge in finite time to an equilibrium point *different* from the origin. Consequently, there is a structural obstruction for sign controllers application for the systems of the order three and higher. The main contribution of this paper is to present an alternative way to design a 3-sign controller that makes the trajectories of a triple integrator converge to the origin. The work is organized as follows. Section II includes the notation used throughout this paper. In Section III we present a Lyapunov based methodology to check the stability of a class of arbitrary order systems dependent on signs of the state variables. This class of systems is composed by a linear feedback of all the state variables, and their signs. The stability proof is endowed with linear matrix inequalities (LMIs). This result is used in Section IV to prove the convergence of the trajectories of an third-order integrator chain to the origin, when it is in closed loop with a sign controller of the same order as the plant. The difference of this controller with the ones mentioned before is that the control law does not depend on the signs of the state variables, but on the sign of a set of linear functions of them. These functions are determined by a state transformation defined in the same section. The main results are gathered in a theorem that establishes the conditions to ensure the convergence to the origin when the solutions of the system have their initial conditions inside a region of the third dimensional space, followed by a corollary that establishes a condition to guarantee the global asymptotic convergence. In the same section it is considered the case when a measurement of the state is available and thus, a controller that contains a linear feedback is possible. An academic example is provided in Section V, along with some numerical simulations and a discussion of the possibility of achieving a finite-time convergence of the system's solutions. Finally, Section VI contains some conclusions of this work.

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II. Preliminaries

A. Notation

- The element-wise application of an operator $\bullet$ to a vector x is indicated by $\vec{\bullet}(x)$;
- I_n denotes the Identity matrix of n dimension;
- $\mathrm{diag}(A)$ represents a matrix where the main diagonal is the same as that of matrix A and every other element is

equal to zero;
- $\vec{a}_{(n\times s)}$, for a constant a represents a matrix of dimension $n\times s$, whose every element is equal to a;
- $A^{[i]}$ denotes the ith column of a matrix A;
- A_n^{int} represents a square matrix of size n whose every element is equal to zero, except for the diagonal above the main one, which is composed of ones;
- $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimal and the maximal eigenvalues of a matrix A, respectively;
- $|a|$ represents the absolute value of a scalar a;
- $\|v\|$ denotes the Euclidean norm of a vector $v\in\mathbb{R}^n$, $\|v\|_1=\sum_{i=1}^n|v_i|$ and $\|v\|_\infty=\max_{1\le i\le n}|v_i|$;
- $\|A\|$ represents the induced norm of a matrix $A\in\mathbb{R}^{n\times s}$, while $\|A\|_1=\max_{1\le i\le s}\|A^{[i]}\|_1$ and $\|A\|_\infty=\|A^T\|_1$;
- For a matrix $A\in\mathbb{R}^{n\times s}$ the following norm equivalences hold [3]
$$\frac{1}{\sqrt{n}}\|A\|_1\le\|A\|\le\sqrt{s}\|A\|_1. \tag{1}$$
- The set of all functions endowed with the (essential) supremum norm $\|w\|_\infty=(\mathrm{ess})\sup_{t\ge0}\|w(t)\|\le W<\infty$, is denoted by L_∞^m

III. Stability of an n-dimensional system with n signs

A stability proof for a class of n-dimensional systems, whose dynamics consist of the sum of a purely linear part and the signs of the state variables, will be developed in this section. This kind of systems can be written in the following form
$$\dot{x}=A_0x+A_1\vec{\mathrm{sign}}(x), \tag{2}$$
where $x\in\mathbb{R}^n$ is the state vector, $\vec{\mathrm{sign}}(x)\in\mathbb{R}^n$ is a column defined as $\vec{\mathrm{sign}}^T(x):=\begin{bmatrix}\mathrm{sign}(x_1) & \dots & \mathrm{sign}(x_n)\end{bmatrix}$ and $A_0,A_1\in\mathbb{R}^{n\times n}$ are real constant matrices. The stability check will be performed by establishing a sufficient LMI condition to construct a Lyapunov function for (2). To this end, the matrices $P=P^T\in\mathbb{R}^{n\times n}$, $G\in\mathbb{R}^{n\times n}$, $R\in\mathbb{R}^{n\times n}$, $r\in\mathbb{R}^{n\times n}$ and $M\in\mathbb{R}^{n\times n}$, and a constant μ must be defined in the following manner:
$$G:=\begin{bmatrix}g_1 & 0 & \dots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & g_n\end{bmatrix}\qquad R:=2A_1^TP+GA_0, \tag{3}$$
$$r:=\mathrm{diag}(R),\quad M:=r+|R\vec{-}r|,\quad \mu:=\max\{M\vec{1}_{(n\times1)}\}$$
for some real constants g_i, $i=1,\dots,n$. The following theorem establishes the LMI conditions that the above defined parameters should satisfy in order to construct a Lyapunov function for (2).

Theorem 1: Let the origin be the only equilibrium of (2), and the following pair of LMIs be satisfied
$$A_0^TP+PA_0=-Q,\qquad M\vec{1}_{(n\times1)}\le0,$$
for
$$P=P^T>0,\qquad Q=Q^T>0,\qquad G\ge0,\qquad GA_1=0.$$
Then, a function $V:\mathbb{R}^n\to\mathbb{R}$ defined as
$$V(x):=x^TPx+\sum_{i=1}^n g_i|x_i| \tag{4}$$
is a Lyapunov function for system (2), with a derivative estimate
$$\dot{V}\le-x^TQx.$$

Proof Since the matrix P is positive definite and G is positive semidefinite, then V is also positive definite and radially unbounded. The function V is locally Lipschitz continuous, then by Rademacher's theorem it is differentiable almost everywhere in $\mathbb{R}^n$, and the derivative of V along the trajectories of (2) is
$$\begin{aligned}\dot{V} &= x^T(A_0^TP+PA_0)x+\vec{\mathrm{sign}}^T(x)(A_1^TP+\frac{1}{2}GA_0)x\\ &\quad+x^T(PA_1+\frac{1}{2}A_0^TG)\vec{\mathrm{sign}}(x)\\ &= x^T(A_0^TP+PA_0)x+\vec{\mathrm{sign}}(x)^TRx.\end{aligned} \tag{5}$$
The Lyapunov equation $A_0^TP+PA_0=-Q$ can be solved for $P=P^T>0$, with $Q=Q^T>0$ if and only if A_0 is a stable matrix. From the second part of (5) we have that
$$\begin{aligned}\vec{\mathrm{sign}}(x)^TRx &= \sum_{i=1}^n\left[x_i\sum_{k=1}^n\left(\mathrm{sign}(x_k)R_{k,i}\right)\right]\\ &\le\sum_{i=1}^n\left[\left(R_{i,i}+\sum_{k\ne i,k=1}^n|R_{k,i}|\right)|x_i|\right]\end{aligned}$$
and with $M=r+|R\vec{-}r|$, as defined above, we get
$$R_{(i,i)}+\sum_{k\ne i}^n|R_{(k,i)}|\le0,\ \forall1\le i\le n\Leftrightarrow M^T\vec{1}_{(n\times1)}\le0.$$
Thus,
$$\dot{V}\le-x^TQx,$$
and if the conditions of Theorem 1 are satisfied, the function V is positive definite for all x, and its derivative along the trajectories of (2) is negative definite.

IV. Stability of a triple integrator governed by a 3-sign controller

One of the canonical representations of a linear system's dynamics is through a chain of integrators with a control input u in the last channel. For the third order that would be
$$\dot{x}_1=x_2,\qquad\dot{x}_2=x_3,\qquad\dot{x}_3=u. \tag{6}$$
As mentioned in the introduction, the stabilization of such system using only the signs of the states variables, *i.e.* when the control input is defined as
$$u=-k_1\mathrm{sign}(x_1)-k_2\mathrm{sign}(x_2)-k_3\mathrm{sign}(x_3), \tag{7}$$
is a problem that has interested the control community for a long time. Numerous efforts have been dedicated to solve this problem, even for different orders, although the global convergence of the solutions to the origin has only been proven for orders one and two. It is easy to see that this

closed loop would represent a special case of the system (2) when $A_0 = A_3^{int}$ and A_1 is composed of only zeros except for its the last row, which contains the gains of the controller $-k_1$, $-k_2$ and $-k_3$. In the following paragraphs we will present a way of designing a 3-sign controller for the chain of integrators (6) that depends on some linear combinations of the state x, *i.e.*

$$u = -k_1 \mathrm{sign}(f_1(x)) - k_2 \mathrm{sign}(f_2(x)) - k_3 \mathrm{sign}(f_3(x)). \quad (8)$$

It will be shown how to construct each of the functions f_1, f_2, f_3 exactly from the available state measurements. By the introduction of a state transformation, it will be shown that system (6) with the above mentioned controller is also equivalent to a special form of (2) and thus, the result presented in the previous section can be used to prove the stability of the closed loop. That means that the trajectories of the third order integrator can be proved to converge to the origin, only using a triple sign controller.

Remark 1: It has been mentioned that the triple integrator in closed loop with the controller (7) is a special case of the more general system (2), as well as the closed loop of (6) and (8) when a state transformation is applied. Therefore, through a similar procedure, the approach presented in the last section could be applied to other classes of systems for which a suitable transformation is found.

A. State transformation to a new set of coordinates

For the chain of integrators (6) let's define the state transformation $z := Tx$ with invertible T given by

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad (9)$$

In the new coordinates the system has the form

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= -z_2 + z_3 \\ \dot{z}_n &= \bar{C}^T z + u, \end{aligned}$$

where $\bar{C} := \left((TA_3^{int}T^{-1})^T\right)^{[3]} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T$. If for the above system the control law is selected as

$$u = -k_1 \mathrm{sign}(z_1) - k_2 \mathrm{sign}(z_2) - k_3 \mathrm{sign}(z_3), \quad (10)$$

and defining vectors B, $C \in \mathbb{R}^3$ as

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 + 1 \end{bmatrix}, \quad (11)$$

it can be written in the form

$$\dot{z} = A_0 z + A_1 \vec{\mathrm{sign}}(z) + BC^T z, \quad (12)$$

where

$$A_0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -k_1 & -k_2 & -k_3 \end{bmatrix}. \quad (13)$$

Remark 2: Note that each variable z_1, z_2, z_3 is equivalent to a linear combination of x_1, x_2, and x_3 of (6) given by the transformation $z := Tx$, so the functions f_1, f_2 and f_3 of controller (8) should be constructed as

$$\begin{aligned} f_1(x) &= x_1 \\ f_2(x) &= x_1 + x_2 \\ f_3(x) &= x_1 + 2x_2 + x_3. \end{aligned}$$

If $C = 0$ in (12), then the transformed system corresponds to a special form of (2), and the term $BC^T z$ can be interpreted as a Lipschitz perturbation.

B. Convergence to the origin of a 3-signs controller

In the previous section a new set of coordinates z was defined. Note that each of the state variable z_1, z_2, z_3, is equivalent to a linear combination of the states x of (6), given by $z_1 = x_1$, $z_2 = x_1 + x_2$, and $z_3 = x_1 + 2x_2 + x_3$. The following theorem uses this set of coordinates to define a 3-sign controller for the chain of integrators (6), and also establishes conditions to guarantee the convergence to the origin of the closed-loop system.

Theorem 2: If for the chain of integrators (6) a control law is selected as (10), then every solution of the closed loop starting in the set

$$\Omega = \left\{ z \in \mathbb{R}^3 : \|z\| < \kappa \right\},$$

where

$$\kappa := \frac{1}{2\lambda_{\max}(P)} \left[\sqrt{3 g_{\max}^2 + 4 \lambda_{\max}(P) \lambda_{\min}(P) \frac{\mu^2}{\alpha^2}} - \sqrt{3} g_{\max} \right],$$
$$\alpha := \lambda_{\min}(Q) - 2\lambda_{\max}(P) \vec{\max}\{C\},$$
$$g_{\max} := \max\{g_1, g_2, g_3\}$$

and the matrices P, Q, $G = \mathrm{diag}(g)$ and $\mu := \max\{M\vec{1}_{(3\times 1)}\}$ come from Theorem 1 for the nominal system (2), will asymptotically converge to the origin provided that the constants k_1, k_2, and k_3 are strictly positive and they satisfy

$$\mu < 0, \quad k_1 < k_3.$$

Corollary 1: If, additionally,

$$\max\{C\} \leq \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},$$

then every solution of the closed-loop system will asymptotically converge to the origin.

Proof The term $BC^T z$ in (12) can be considered as a perturbation to the nominal one (2). Therefore, first, the stability of the system for the case $C = 0$ will be shown and, second, the set of invariance for $C \neq 0$ will be evaluated. For the first part of the proof we will check the existence of a unique equilibrium at the origin for the nominal system. To this end the matrix A_0 can be divided as $A_0 = \begin{bmatrix} A_0^u \in \mathbb{R}^{2\times 3} \\ A_0^l \in \mathbb{R}^{1\times 3} \end{bmatrix}$, where the last line has been separated since the signs appear for $\dot{z}_3$ only. Now, let us define a column vector $\beta^T = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$, representing the coordinates of a possible equilibrium, and look for the values of β_i, $i=1,2,3$ which annihilate A_0^u, that is

$$A_0^u\beta = \begin{bmatrix}-\beta_1+\beta_2\\ -\beta_2+\beta_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

For the above to hold, either

$$\beta_3 = 0 \Rightarrow \beta_1=\beta_2=0 \tag{14}$$

or

$$\beta_3 \neq 0 \Rightarrow \beta_1 = -\beta_2 \neq 0. \tag{15}$$

In the case of (14), the equilibrium point is at the origin. The case of (15) implies that all signs in the last equation for $\dot{z}_3$ are some constants and, under the theorem conditions, in such a case $\dot{z}_3 \neq 0$. Therefore, it has been proven that the only equilibrium point is at the origin. Now we will check the boundedness, the region of attraction and stability of the solutions of the nominal system. To this end, consider the Lyapunov function (4) satisfying $A_0^T P + PA_0 < -Q$, $P=P^T>0$, $Q=Q^T>0$, and $GA_1=0$ (see Theorem 1 for the details). From the following expression,

$$\lambda_{\min}(P)\|z\|^2 \leq V(z) \leq \lambda_{\max}(P)\|z\|^2 + g_{\max}\|z\|_1,$$

we can see that since P is positive definite, $V(z)>0\ \forall\ x\neq 0$, and that the function is radially unbounded, *i.e.* $V(z)\to\infty$ as $\|z\|\to\infty$. Using inequality (1), the derivative of (4) along the nominal part of (12) can be expressed as

$$\begin{aligned}\dot{V} &= -z^T Qz + \vec{\text{sign}}^T(z)Rz\\ &\leq -\lambda_{\min}(Q)\|z\|^2 + \mu\|z\|_1,\end{aligned}$$

where μ has been defined in (3). Since $\mu<0$, which is equivalent to establishing a strict inequality for the condition on M in Theorem 1, the derivative of the Lyapunov function can be expressed as

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(Q)\|z\|^2 - |\mu|\|z\|_1\\ &\leq -\lambda_{\min}(Q)\|z\|^2 - |\mu|\|z\|.\end{aligned}$$

It is evident that in this case, $\dot{V}<0$ for all $z\neq 0$. From section III we have that $M = r + |R\vec{-}r|$, so μ is equal to the maximum of all the column sums of M. Since A_0 is a Metzler matrix, then the first condition of theorem 1 can be solved for a diagonal P, then M, for $k_1,k_2,k_3>0$ has the form

$$M = \begin{bmatrix}-g_1 & g_1 & -2k_1P_{3,3}\\ 0 & -g_2 & g_2-2k_2P_{3,3}\\ 0 & 0 & -2k_3P_{3,3}\end{bmatrix}.$$

The last element of the product $M^T\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ is

$$|g_2 - 2k_2P_{3,3}| + 2k_1P_{3,3} - 2k_3P_{3,3}.$$

Choosing $g_2 = 2k_2P_{3,3}$, μ can be made negative by satisfying

$$k_1 < k_3$$

and $0<g_1<g_2$. This also satisfies the LMI restriction $G\geq 0$ of Theorem 1. The last restriction, $GA_1=0$ is achieved by simply choosing $g_3=0$. Thus, the nominal system (2) with $C=0$ has the only equilibrium at the origin, and LMIs of Theorem 1 are satisfied, then the system is globally asymptotically stable. For the perturbed case, when $C\neq 0$, assuming that $\mu<0$ the derivative of (4) along (12) is

$$\begin{aligned}\dot{V}(z) &= -z^TQz + z^T\left(CB^TP+PBC^T\right)z\\ &\quad +\vec{\text{sign}}^T(z)Rz + \vec{\text{sign}}^T(z)GBC^Tz\\ &\leq -\lambda_{\min}(Q)\|z\|^2 + 2\lambda_{\max}(P)\vec{\max}\{C\}\|z\|^2\\ &\quad -|\mu|\|z\|_1\\ &\leq \alpha\|z\|^2 - |\mu|\|z\|.\end{aligned}$$

Note that in the above expression, the term $GBC^T=0$, since by construction $GA_1=0$ and BC^T has the same structure as A_1. First, consider the case when $\alpha>0$, and define the set

$$\Omega_1 := \left\{z\in\mathbb{R}^3 : \dot{V}\leq 0\right\} = \left\{z\in\mathbb{R}^3 : \|z\|\leq\frac{|\mu|}{\alpha}\right\},$$

and $\dot{V}<0$ in the interior of the set Ω_1 for all $z\neq 0$. To find an invariant set inside Ω_1, recall the definition of V from which we have that

$$\lambda_{\min}(P)\|z\|^2 \leq V(z) \leq \lambda_{\max}(P)\|z\|^2 + \sqrt{3}g_{\max}\|z\|,$$

and note that in order to have $z\in\Omega_1$ the following inequality has to be satisfied:

$$V(z) \leq \lambda_{\min}(P)\frac{\mu^2}{\alpha^2},$$

which is true if

$$\lambda_{\max}(P)\|z\|^2 + \sqrt{3}g_{\max}\|z\| \leq \lambda_{\min}(P)\frac{\mu^2}{\alpha^2}.$$

Solving the last inequality with respect to $\|z\|$ we obtain

$$\|z\|\leq\kappa,$$

then $\Omega = \{z\in\mathbb{R}^3 : \|z\|<\kappa\}\subset\Omega_1$ and every solution starting in Ω will remain inside in Ω for all $t\geq 0$, in addition

$$\dot{V}<0 \quad \forall z\in\Omega\setminus\{0\},$$

therefore, all trajectories will converge to the origin for the initial conditions in Ω. Theorem 2 is proven. For the case when $\alpha\leq 0$, it is easy to see that $\dot{V}(z)\leq 0$ for all z, and $\dot{V}(z)=0$ only at the origin, so in this case the perturbed system is globally stable. For this, the following inequality has to be satisfied

$$\max\{C\} \leq \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},$$

which proves Corollary 1.

C. Convergence to the origin of a 3-sign controller with an added linear term

In certain control application it is possible that the state of a system is available for measurements and thus, the design of a controller that contains a linear feedback is also interesting. This section will explore this case, and consider a control law that includes the signs of the functions defined in the previous section, and also a linear combination of the states z (and, hence, of x also). Such controller is defined in the following theorem, which also establishes the conditions for its design that guarantee the convergence of all solutions of (6) to the origin.

Theorem 3: If for the chain of integrators (6) a control law is selected as

$$u = -k_1 \mathrm{sign}(z_1) - k_2 \mathrm{sign}(z_2) - k_3 \mathrm{sign}(z_3) - Cz,$$

then every solution of the closed-loop system will converge to the origin asymptotically provided that the constants k_1, k_2, and k_3 are chosen strictly positive, and the latter satisfy

$$\mu < 0, \qquad k_1 < k_3.$$

Proof System (6) in closed loop with the controller of theorem 3 takes the form of system (12) with $C = 0$. Next, the proof follows the arguments demonstrated in the proof of Theorem 2.

V. Example and numerical simulations

For the triple integrator (6), with a matched disturbance, i.e.

$$x_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = u + w, \tag{16}$$

the control law was chosen following the conditions established in Theorem 2, as

$$\begin{aligned} u \;\; = \;\; & -2\mathrm{sign}(x_1) - 2\mathrm{sign}(x_1 + x_2) \\ & -2.5\mathrm{sign}(x_1 + 2x_2 + x_3). \end{aligned} \tag{17}$$

With initial conditions

$$x_1(0) = 0.5,\; x_2(0) = -0.8,\; x_3(0) = 2, \tag{18}$$

and a matched disturbance $w = 1 + 0.5\sin(t)$, the obtained simulation results are shown in Fig. 1, for a sampling step of $0.0001s$. In [7] it was defined a three dimensional invariant

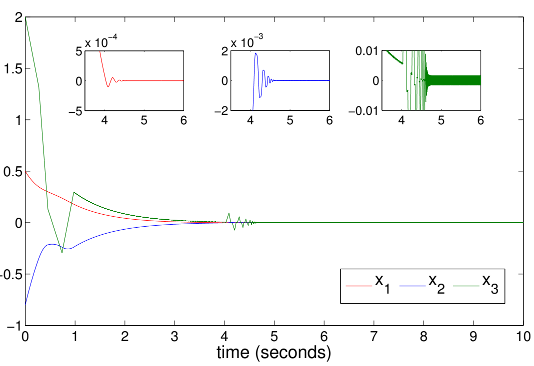

Fig. 1. State trajectories of the triple integrator with the control (17), a matched disturbance with bound $\|w\| = 1$, and the initial conditions (18)

set such that if the initial conditions of the triple integrator are inside it, even with an appropriate set of gains, the trajectories of the system will converge to an equilibrium different from the origin. By simple calculations it can be seen that the trajectories of the 3-integrator chain of this example start precisely inside this set, for the initial conditions in (18) and the selected gains of (17). However, we can see from the simulations, which were ran with a small sampling step and for a short period of time, that the trajectories converge to zero. Moreover, from the zooms it is noticeable that they do this in a finite time even in the presence of a matched disturbance.

Simulations were also ran for the perturbed triple integrator (16) with initial conditions outside of the region of attraction:

$$x_1(0) = 5,\; x_2(0) = 9,\; x_3(0) = -8, \tag{19}$$

in closed loop with the controller (17) with an added linear term, according to Theorem 3 as

$$\begin{aligned} u \;\; = \;\; & -2\mathrm{sign}(x_1) - 2\mathrm{sign}(x_1 + x_2) + \\ & -2.5\mathrm{sign}(x_1 + 2x_2 + x_3) - x_1 + \\ & +3(x_1 + x_2) - 3(x_1 + 2x_2 + x_3), \end{aligned} \tag{20}$$

with the same $w = 1 + 0.5\sin(t)$. The obtained simulation results are shown in Fig. 3, for a sampling step of $0.0001s$. Also the case of a triple integrator with matched and un-

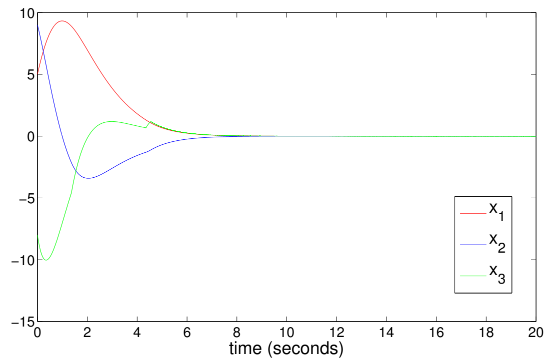

Fig. 2. State trajectories of the triple integrator with the control (20), an matched disturbance, and the initial conditions (19)

matched disturbances was tested via simulations, i.e, $\dot{x}_1 = x_2 + w_u, \dot{x}_2 = x_3, \dot{x}_3 = u + w$, with the same w as in the above example, and $w_u = 0.5\sin(t)$, the obtained simulation results are shown in Fig. 3, for a sampling step of $0.0001s$. In this

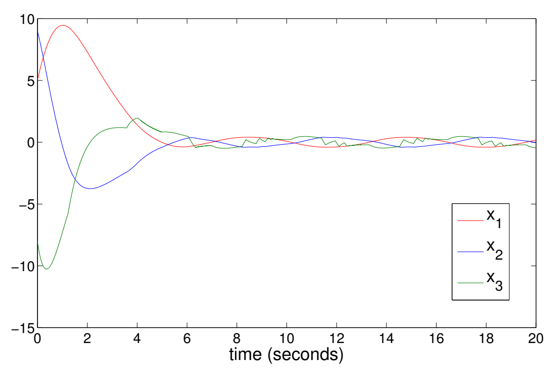

Fig. 3. State trajectories of the triple integrator with the control (20), an unmatched disturbance, and the initial conditions (19)

case, it can be observed that the convergence to the origin is maintained for large initial conditions, and that the matched perturbation is still rejected. Even more, in the presence of an unmatched disturbance, a bounded-input-bounded-state (BIBS)-like behavior can be appreciated.

VI. CONCLUSIONS

For the well studied system of triple integrator, we have presented a 3-sign controller that forces the trajectories to the origin, which is the main contribution of this paper. The main characteristic of this controller is that it does not depend on the sign of the state variables, as the ones that have appeared several times in the control literature, but on the signs of a set of linear functions of them. This result has been achieved by the development of some other preliminary ones. For example, we have proved that the only equilibrium point of a more general class of non-linear systems is at the origin, and we have presented a transformation by which it is shown that the closed loop of a sign controller with a chain of three integrators is a special case of this class of systems. This transformation also determines the form of the set of functions on which the control law depends. The stability of the mentioned class of systems has been proved by the Lyapunov method and LMIs. The design conditions, that must be satisfied in order to guarantee that the closed loop of the triple integrator converges to the origin have been established in theorems and their proofs, using the same Lyapunov function as before. When a measurement of the state is available for feedback, then a control law that includes a linear part can be considered, and we have shown that in this case, the region of attraction of the closed loop is the whole third dimensional space. In order to illustrate the results, an example has been provided which considers a triple integrator in the case where no information, other than the sign of the transformed coordinates is available. The shown simulations are consistent with the results previously obtained. Moreover, even through the proofs we guarantee only asymptotic convergence to the origin, these simulations show evidence that the convergence is actually achieved in a finite time. Also, for the case where a linear feedback is possible, simulation results show that the convergence is maintained for rather large initial conditions. Moreover, even though the theoretical results do not include this, the matched perturbation is rejected in all cases, and a BIBS-like behavior can be noticed when unmatched perturbations are present. It is a direction of our future research to investigate these finite-time convergence and BIBS properties in order to be able to characterize them in a formal mathematical way. Another direction for further investigation would be the extension of these results for higher orders.

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